

# Data structures for 3D Meshes

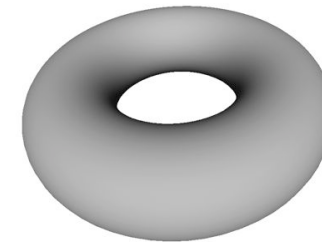
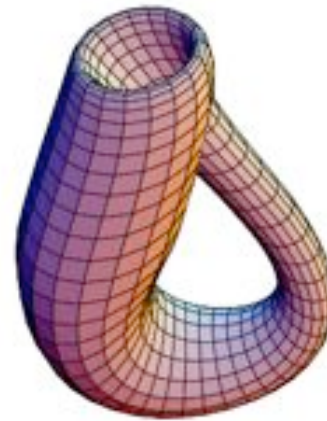
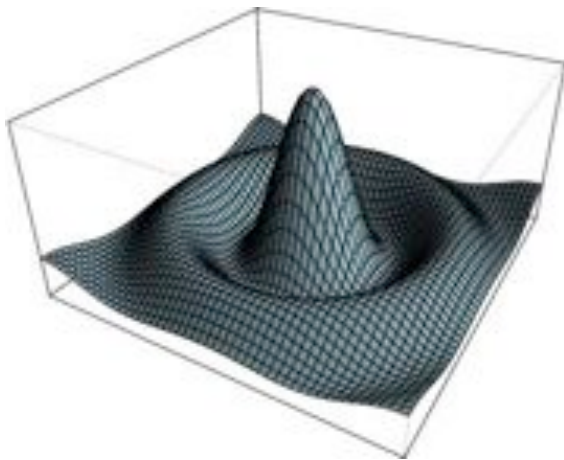
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# Surfaces

- ❖ A 2-dimensional region of 3D space
- ❖ *A portion of space having length and breadth but no thickness*



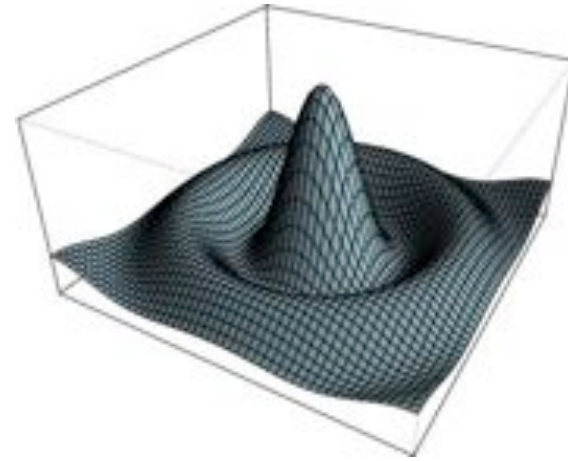
# Defining Surfaces

❖ Analytically...

❖ Parametric surfaces

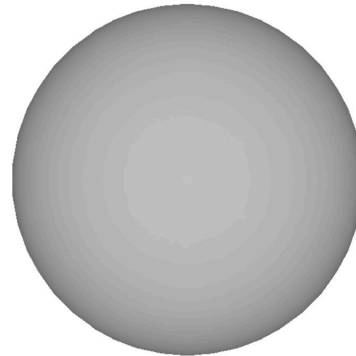
$$S: \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

$$S(x, y) = (x, y, \sin(\sqrt{(x^2 + y^2)})/\sqrt{(x^2 + y^2)})$$

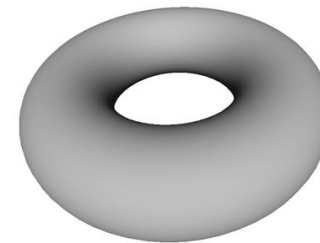


❖ Implicit surfaces

$$S = \{(x, y, z): x^2 + y^2 + z^2 - r^2 = 0\}$$



$$S = \{(x, y, z): (x^2 + y^2 + R^2 - r^2)^2 - 4R^2(x^2 + y^2) = 0\}$$

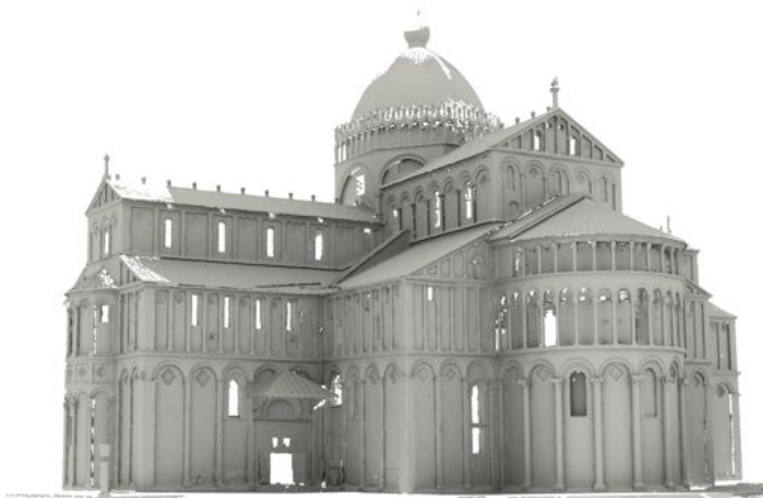


# Representing Real World Surfaces

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- ❖ Analytic definition falls short of representing *real world* surfaces in a *tractable* way

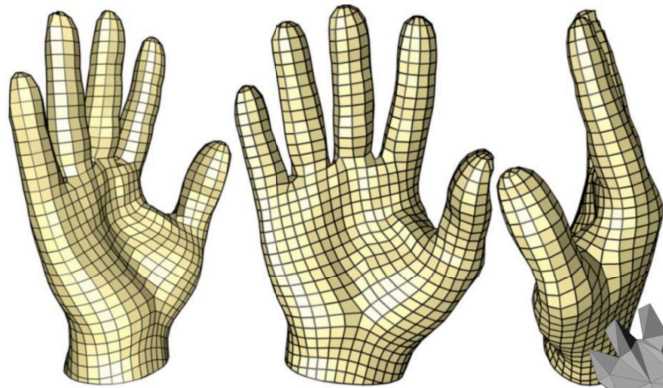
$$S(x, y) = \dots ?$$



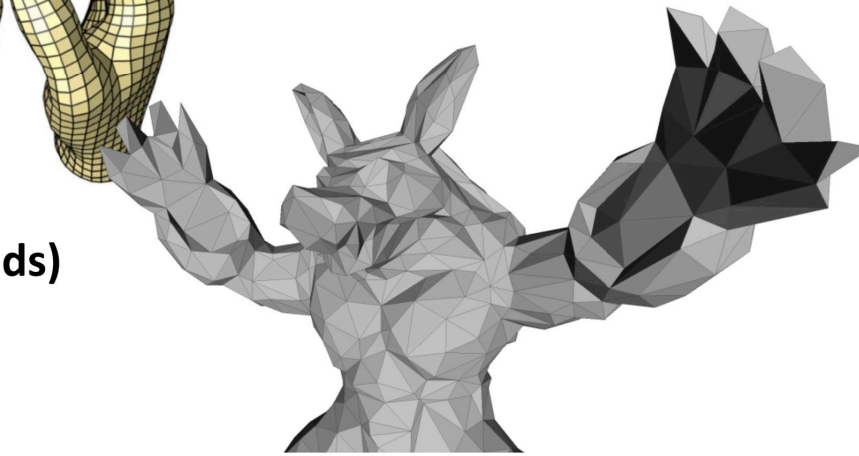
... surfaces can be represented by **cell complexes**

# Cell complexes (meshes)

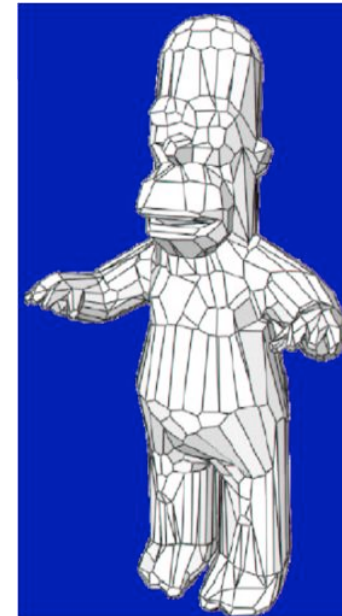
❖ Intuitive description: a continuous surface divided in polygons



quadrilaterals (quads)



triangles



Generic polygons

# Cell Complexes (meshes)

❖ In nature, meshes arise in a variety of contexts:

❖ Cells in organic tissues

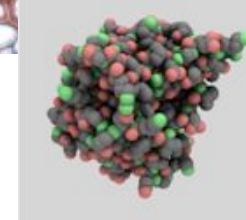
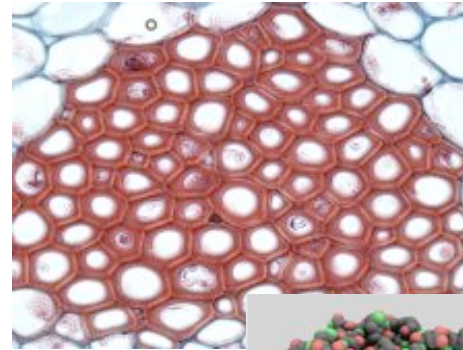
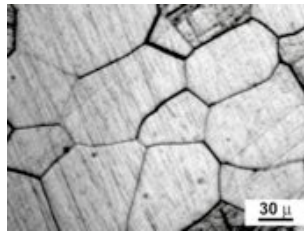
❖ Crystals

❖ Molecules

❖ ...

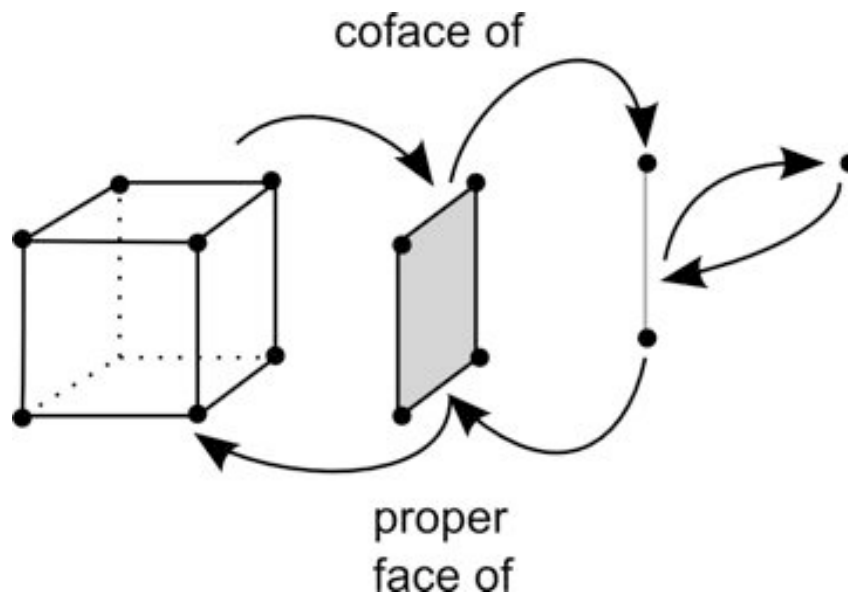
❖ Mostly *convex* but *irregular* cells

❖ Common concept: *complex* shapes can be described as *collections of simple building blocks*



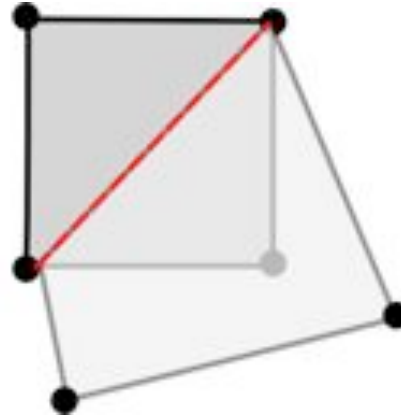
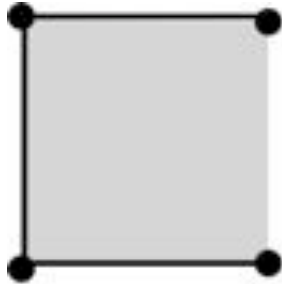
# Cell Complexes (meshes)

- ❖ more formal definition
  - ❖ a *cell* is a convex polytope in
  - ❖ a *proper face* of a cell is a convex polytope in



# Cell Complexes (meshes)

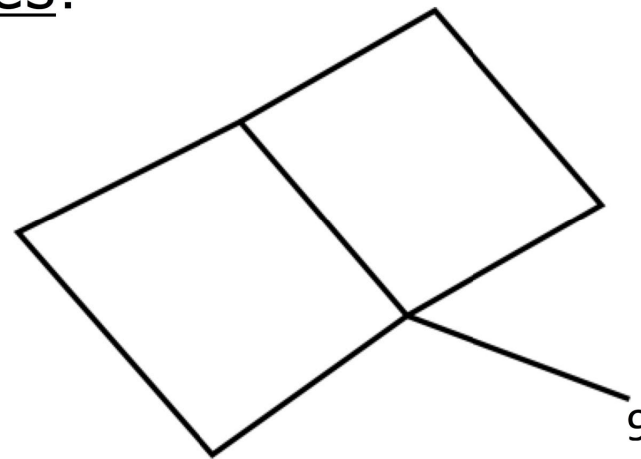
- ❖ a collection of cells is a complex **iff**
  - ❖ every face of a cell belongs to the complex
  - ❖ For every cells  $C$  and  $C'$ , their intersection either is empty or is a common face of both





# Maximal Cell Complex

- ❖ the **order** of a cell is the number of its sides (or vertices)
- ❖ a complex is a **k-complex** if the maximum of the order of its cells is  $k$
- ❖ a cell is **maximal** if it is not a face of another cell
- ❖ a k-complex is **maximal** *iff* all maximal cells have order  $k$
- ❖ short form : no dangling edges!

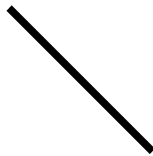


# Simplicial Complex

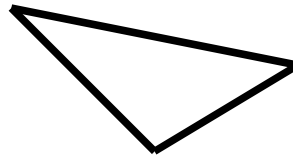
- ❖ A cell complex is a **simplicial complex** when the cells are simplexes
- ❖ A ***d-simplex*** is the convex hull of  $d+1$  points in



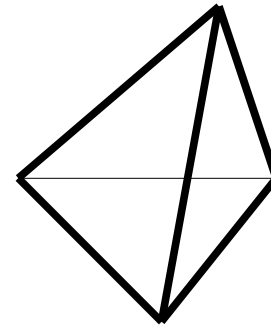
0-simplex



1-simplex



2-simplex

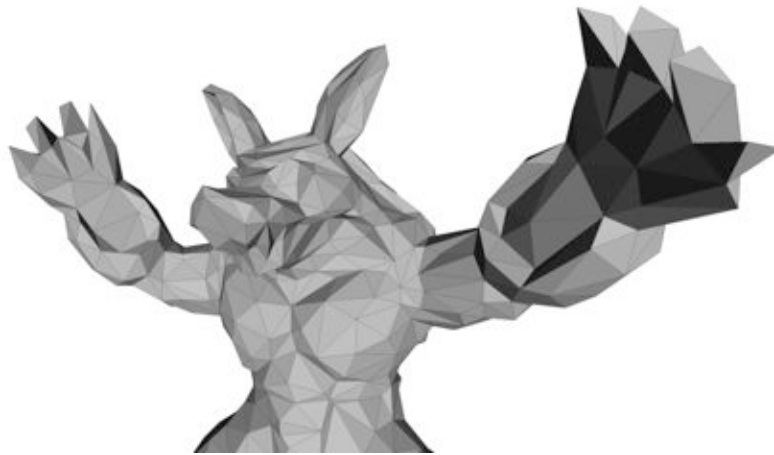


3-simplex

# Meshes, at last

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- ❖ When talking of *triangle mesh* the intended meaning is a **maximal 2-simplicial complex**



# Topology vs Geometry

- ❖ Di un complesso simpliciale e' buona norma distinguere
  - ❖ Realizzazione geometrica
    - ❖ Dove stanno effettivamente nello spazio i vertici del nostro complesso simpliciale
  - ❖ Caratterizzazione topologica
    - ❖ Come sono connessi combinatoriamente i vari elementi

# Topology vs geometry 2

Nota: Di uno stesso oggetto e' possibile dare rappresentazioni con eguale realizzazione geometrica ma differente caratterizzazione topologica (molto differente!) Demo kleine

Nota: Di un oggetto si puo' dire molte cose considerandone solo la componente topologica

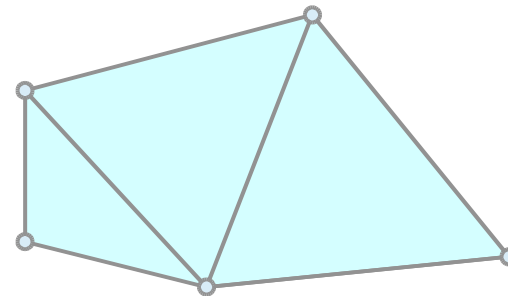
- Orientabilita

- componenti connesse

- bordi

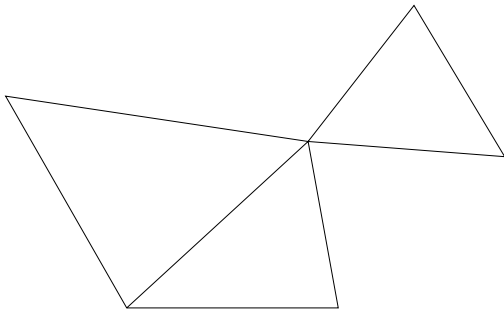
# Incidenza Adiacenza

- ❖ Due semplici  $\sigma$  e  $\sigma'$  sono incidenti se  $\sigma$  è una faccia propria di  $\sigma'$  o vale il viceversa.
- ❖ Due  $k$ -simplessi sono  $m$ -adiacenti ( $k > m$ ) se esiste un  $m$ -simpleso che è una faccia propria di entrambi.
  - ❖ Due triangoli che condividono un edge sono 1-adiacenti
  - ❖ Due triangoli che condividono un vertice sono 0-adiacenti



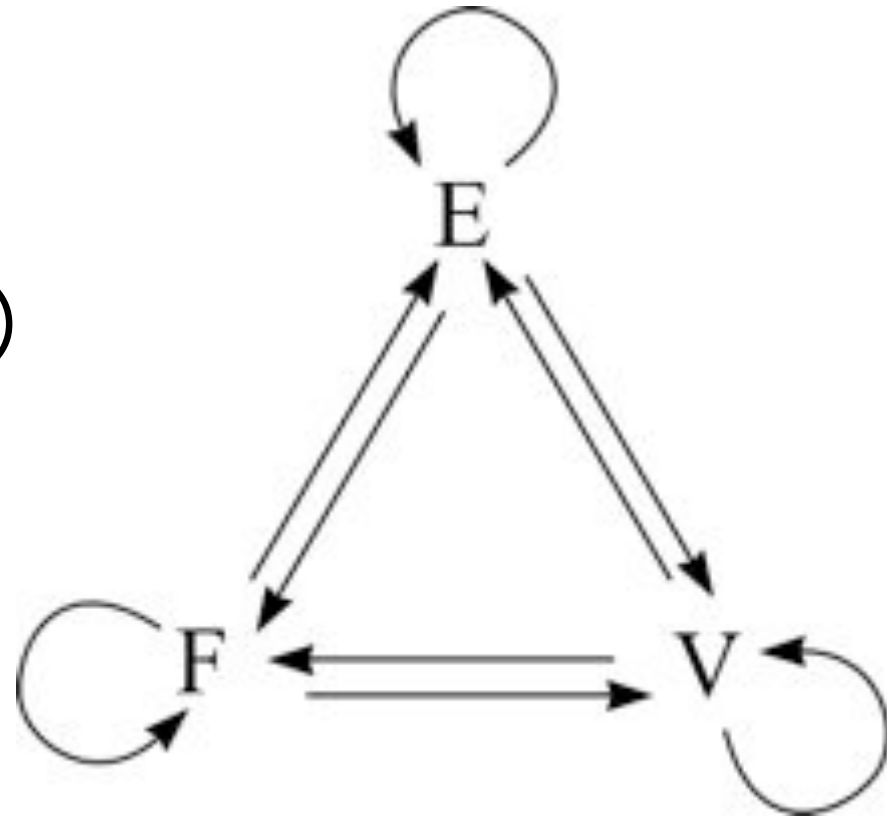
# Relazioni di Adiacenza

- ❖ Per semplicità nel caso di mesh si una relazione di adiacenza con un una coppia (ordinata!) di lettere che indicano le entità coinvolte
  - ❖ FF adiacenza tra triangoli
  - ❖ FV i vertici che compongono un triangolo
  - ❖ VF i triangoli incidenti su un dato vertice



# Relazioni di adiacenza

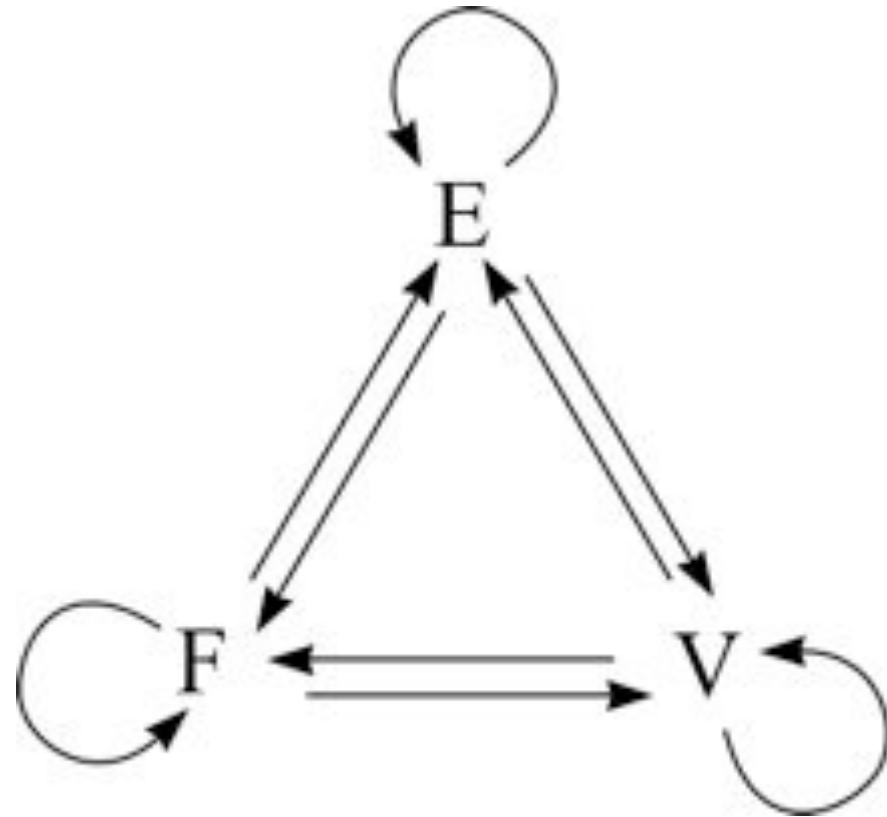
- ❖ Di tutte le possibili relazioni di adiacenza di solito vale la pena se ne considera solo un sottoinsieme (su 9) e ricavare le altre proceduralmente





# Relazioni di adiacenza

- ❖  $FF \sim 1$ -adiacenza
- ❖  $EE \sim 0$  adiacenza
- ❖  $FE \sim$  sottofacce proprie di  $F$  con  $\dim 1$
- ❖  $FV \sim$  sottofacce proprie di  $F$  con  $\dim 0$
- ❖  $EV \sim$  sottofacce proprie di  $E$  con  $\dim 0$
- ❖  $VF \sim F$  in  $\Sigma$  :  $V$  sub faccia di  $F$
- ❖  $VE \sim E$  in  $\Sigma$  :  $V$  sub faccia di  $E$
- ❖  $EF \sim F$  in  $\Sigma$  :  $E$  sub faccia di  $F$
- ❖  $VV \sim V'$  in  $\Sigma$  : Esiste  $E (V, V')$



# Partial adjacency

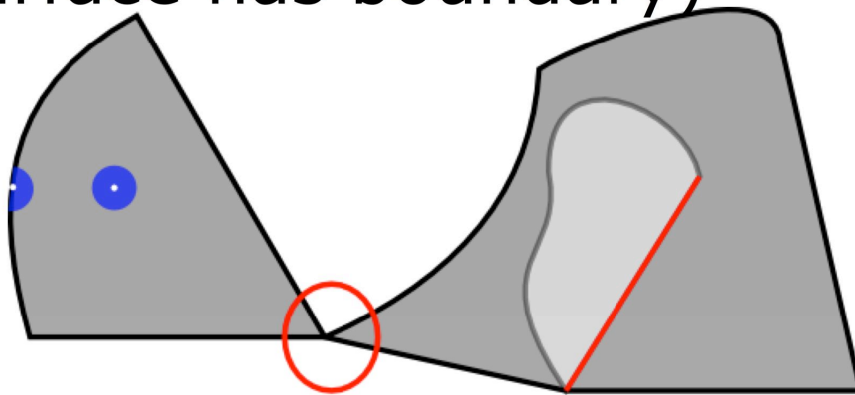
- ❖ Per risparmiare a volte si mantiene una informazione di adiacenza parziale
  - ❖  $VF^*$  memorizzo solo un riferimento dal vertice ad una delle facce e poi 'navigo' sulla mesh usando la FF per trovare le altre facce incidenti su  $V$

# Relazioni di adiacenza

- ❖ In un 2-complesso simpliciale immerso in  $R^3$ , che sia 2 manifold
  - ❖ FV FE FF EF EV sono di cardinalità bounded (costante nel caso non abbia bordi)
    - ❖  $|FV| = 3$   $|EV| = 2$   $|FE| = 3$
    - ❖  $|FF| \leq 2$
    - ❖  $|EF| \leq 2$
  - ❖ VV VE VF EE sono di card. variabile ma in stimabile in media
    - ❖  $|VV| \sim |VE| \sim |VF| \sim 6$
    - ❖  $|EE| \sim 10$
    - ❖  $F \sim 2V$

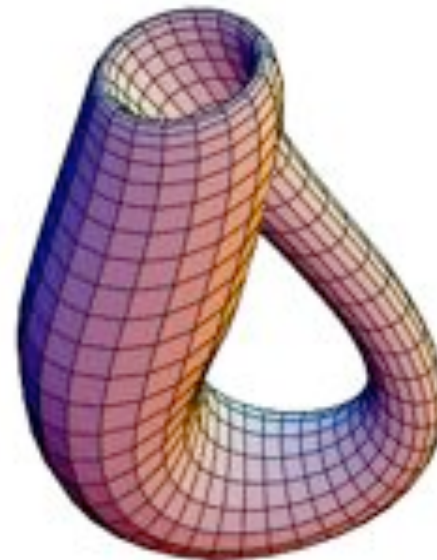
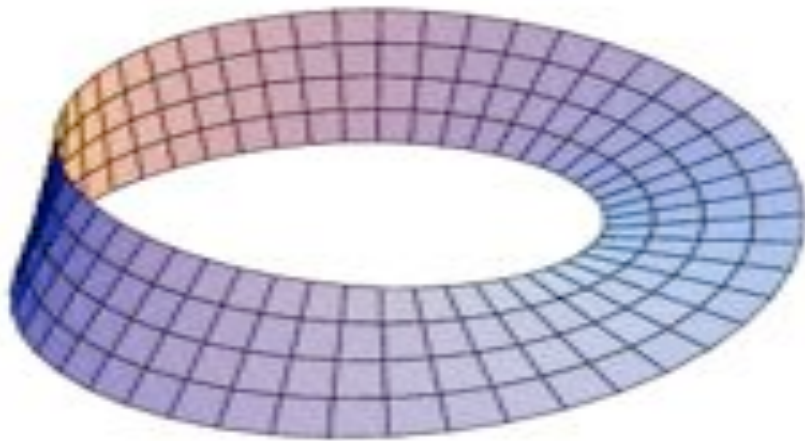
# Manifoldness

- ❖ a surface  $S$  is **2-manifold** *iff*:
  - ❖ the neighborhood of each point is homeomorphic to Euclidean space in two dimension  
*or ... in other words..*
  - ❖ the neighborhood of each point is homeomorphic to a disk (or a semidisk if the surface has boundary)



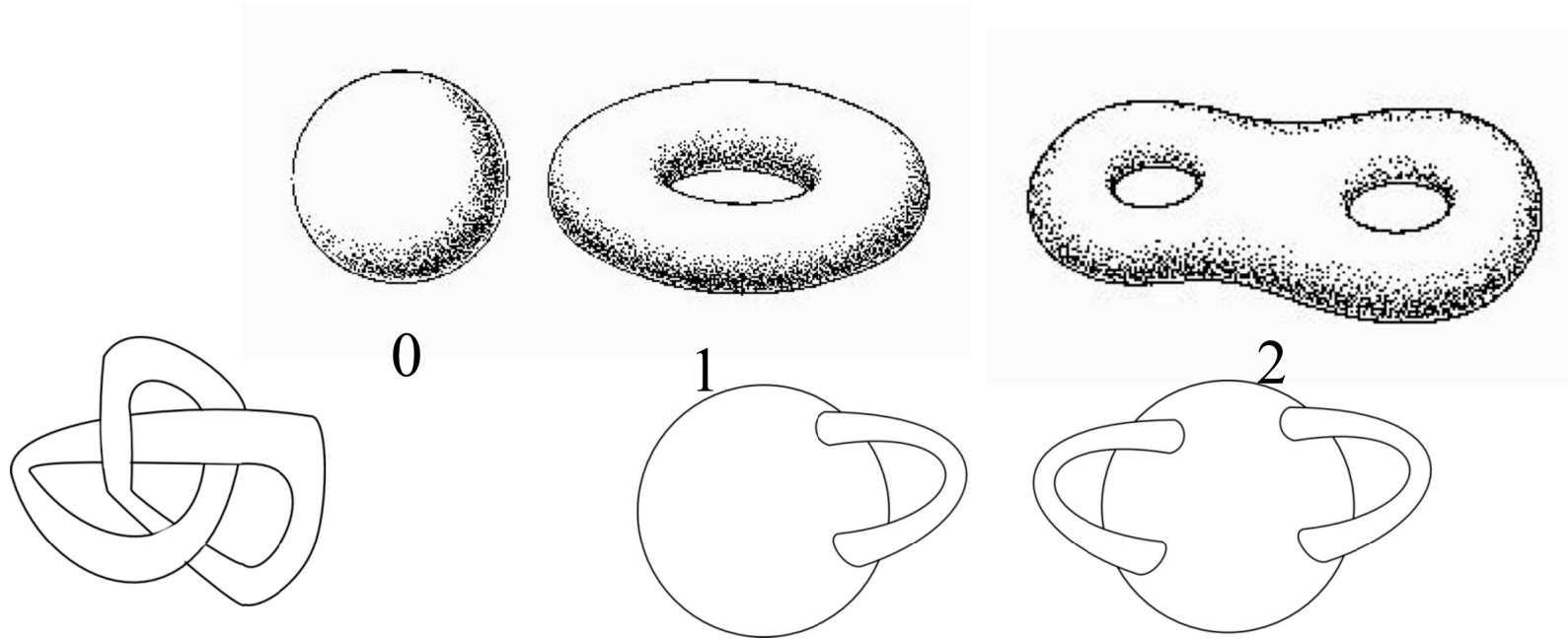
# Orientability

- ❖ A surface is **orientable** if it is possible to make a consistent choice for the normal vector
  - ❖ ...it has two sides
- ❖ Moebius strips, klein bottles, and non manifold surfaces are not orientable



# Genus

❖ The **Genus** of a closed surface, orientable and 2-manifold is the maximum number of cuts we can make along non intersecting closed curves without splitting the surface in two.

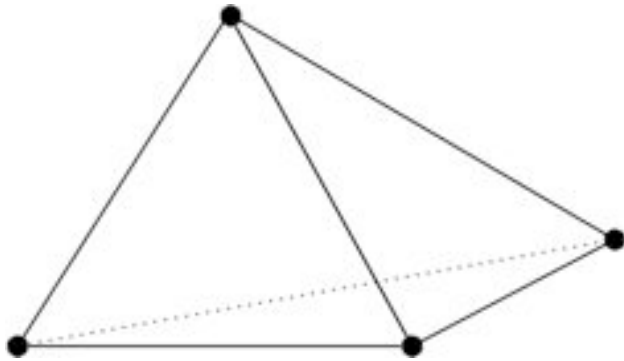


❖ ...also known as the number of *handles*

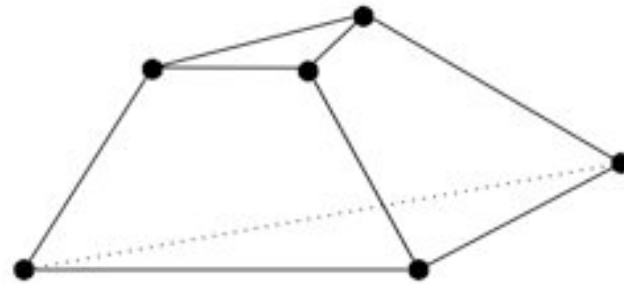
| Name                                               | Image                                                                               | $V$ (vertices) | $E$ (edges) | $F$ (faces) | Euler characteristic: $V - E + F$ |
|----------------------------------------------------|-------------------------------------------------------------------------------------|----------------|-------------|-------------|-----------------------------------|
| <a href="#">Tetrahedron</a>                        |    | 4              | 6           | 4           | 2                                 |
| <a href="#">Hexahedron</a> or <a href="#">cube</a> |    | 8              | 12          | 6           | 2                                 |
| <a href="#">Octahedron</a>                         |    | 6              | 12          | 8           | 2                                 |
| <a href="#">Dodecahedron</a>                       |   | 20             | 30          | 12          | 2                                 |
| <a href="#">Icosahedron</a>                        |  | 12             | 30          | 20          | 2                                 |

# Euler characteristics

- ❖  $\chi = 2$  for any *simply connected* polyhedron
- ❖ proof by construction...
- ❖ play with examples:



$$\begin{aligned}\chi &= V - E + F \\ \chi &= 4 - 6 + 4 = 2\end{aligned}$$

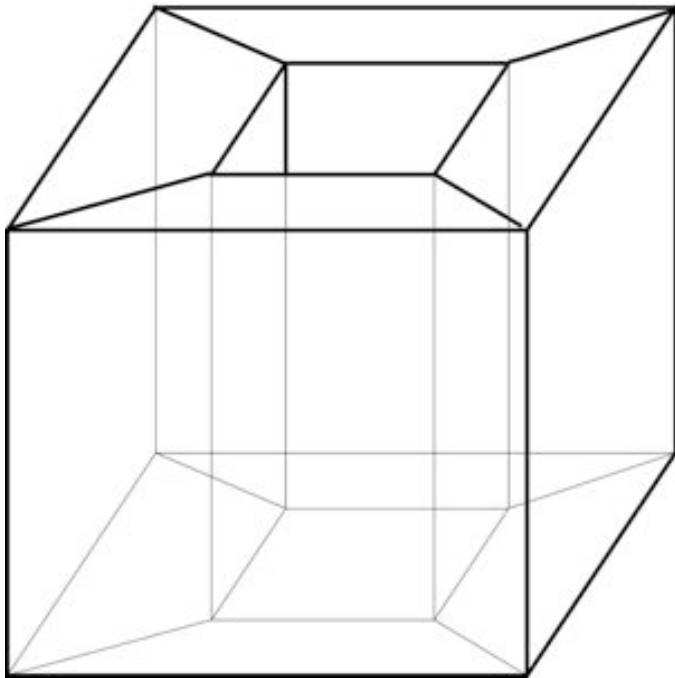


$$\begin{aligned}\chi &= (V + 2) - (E + 3) + (F + 1) = \\ \chi &= (4 + 2) - (6 + 3) + (4 + 1) = 2\end{aligned}$$



# Euler characteristics

❖ let's try a more complex figure...



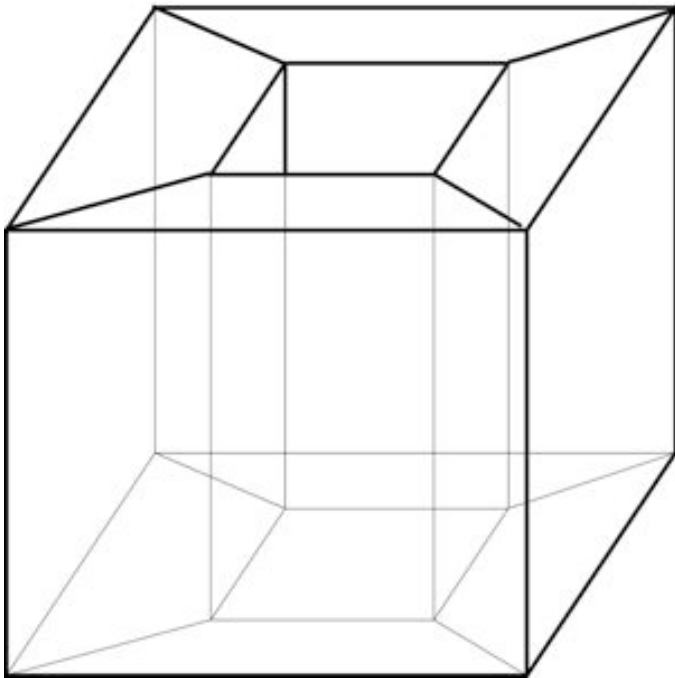
$$\chi = V - E + F$$
$$\chi = 16 - 32 + 16 = \mathbf{0}$$

❖ why = 0 ?

# Euler characteristics

$$\chi = 2 - 2g$$

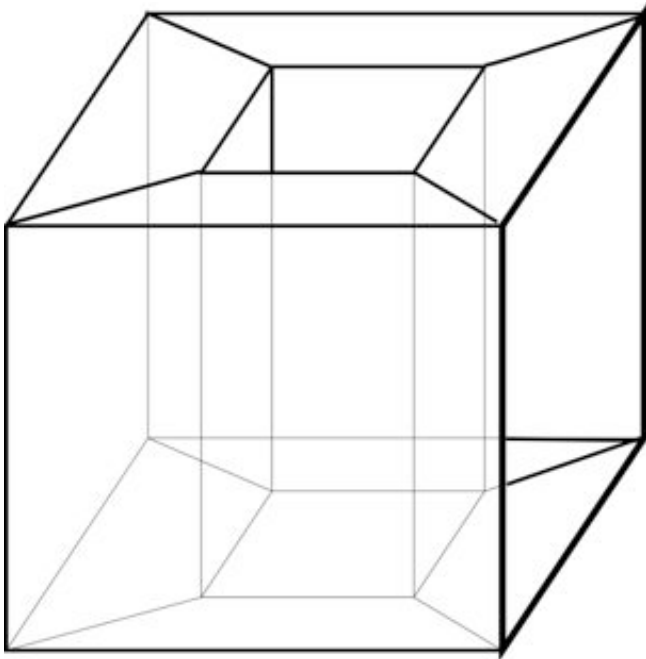
❖ where  $g$  is the genus of the surface



$$\begin{aligned}\chi &= V - E + F \\ \chi &= 16 - 32 + 24 = 0 = 2 - 2g\end{aligned}$$

# Euler characteristics

- ❖ let's try a more complex figure...remove a face. The surface is not closed anymore



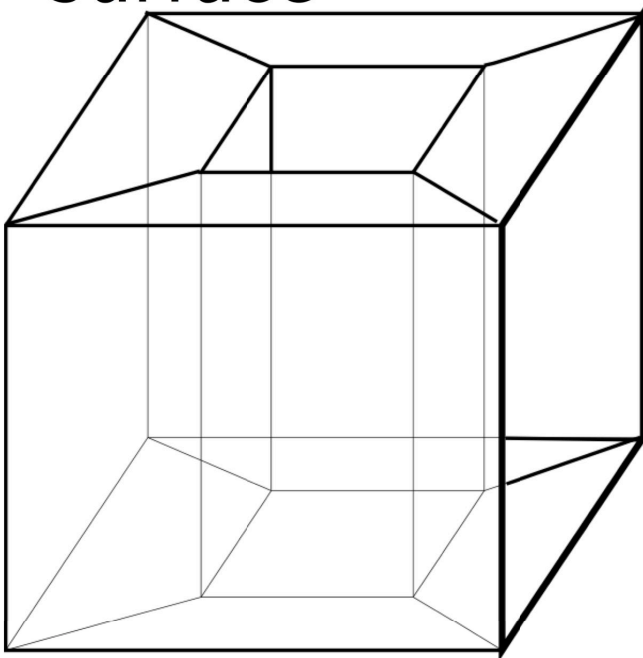
$$\chi = V - E + F$$
$$\chi = 16 - 32 + 15 = -1$$

- ❖ why = -1 ?

# Euler characteristics

$$\chi = 2 - 2g - b$$

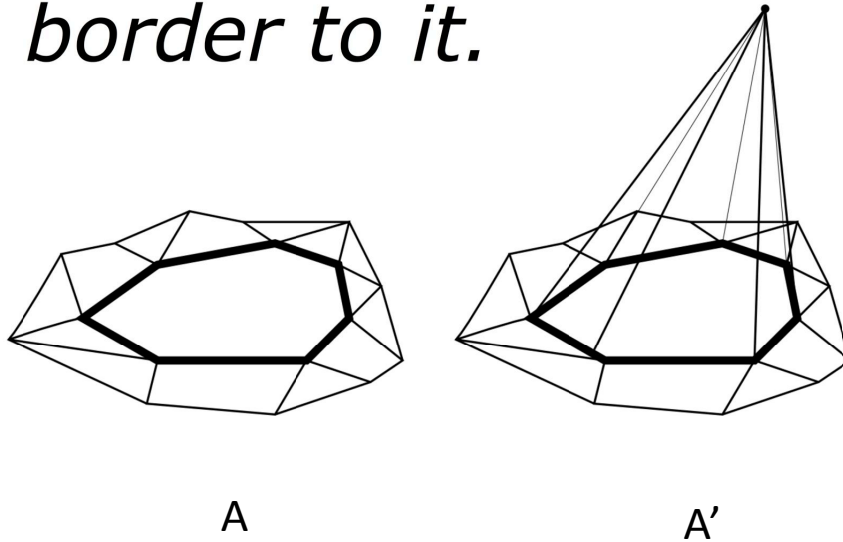
- ❖ where  $b$  is the number of borders of the surface



$$\begin{aligned}\chi &= V - E + F \\ \chi &= 16 - 32 + 15 = 2 - 2g - b\end{aligned}$$

# Euler characteristics

- ❖ *Remove the border by adding a new vertex and connecting all the vertices on the border to it.*



# Differential quantities: normals

- ❖ The (unit) **normal** to a point is the (unit) vector perpendicular to the tangent plane

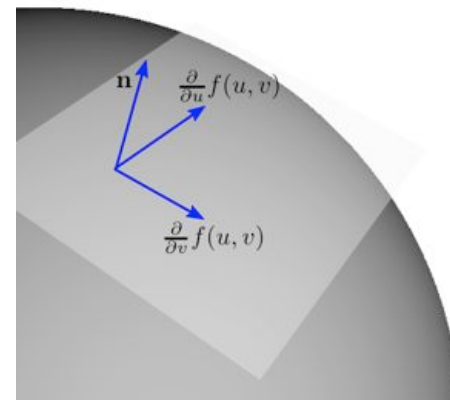
implicit surface:  $f(x, y, z) = 0$

$$n = \frac{\nabla f}{\|\nabla f\|}$$

parametric surface:

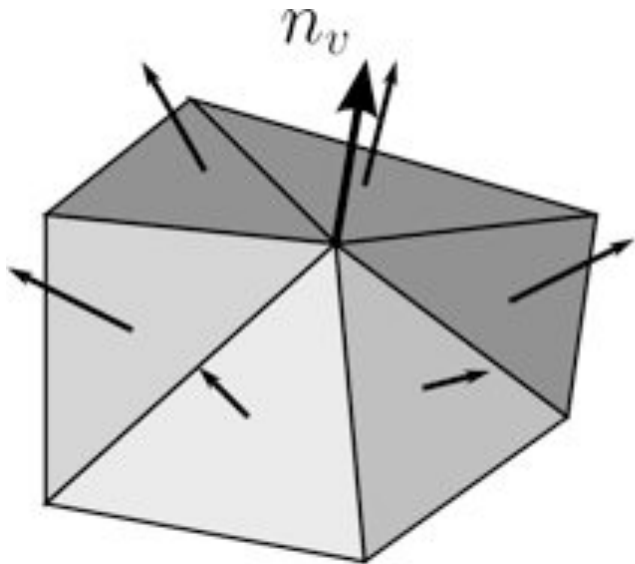
$$f(u, v) = (f_x(u, v), f_y(u, v), f_z(u, v))$$

$$n = \frac{\partial}{\partial u} f \times \frac{\partial}{\partial v} f$$



# Normals on triangle meshes

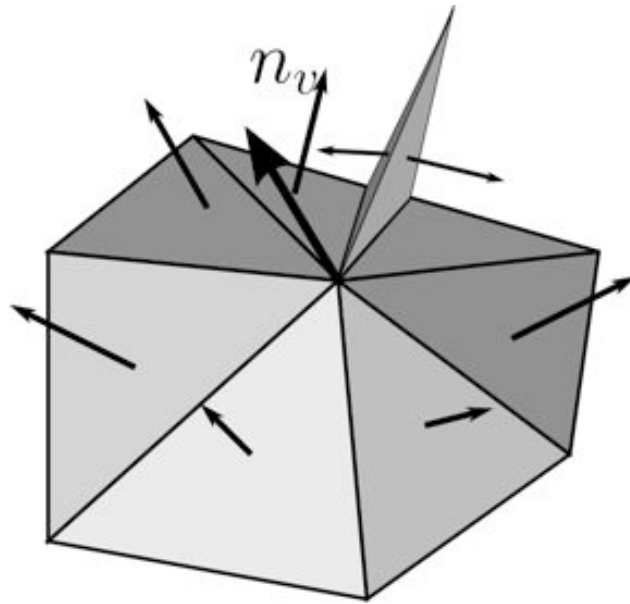
- ❖ Computed per-vertex and interpolated over the faces
- ❖ Common: consider the tangent plane as the average among the planes containing all the faces incident on the vertex



$$n_v = \frac{1}{\#N(v)} \sum_{f \in N(v)} n_f$$
$$N(v) = \{f : f \text{ coface of } v\}$$

# Normals on triangle meshes

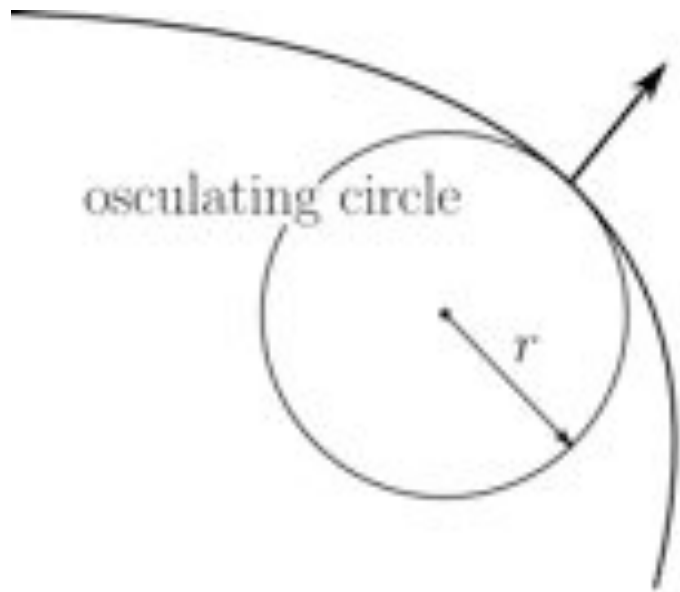
- ❖ Does it work? Yes, for a “good” tessellation
- ❖ Small triangles may change the result dramatically
- ❖ Weighting by edge length / area / angle helps





# Differential quantities: Curvature

- ❖ The curvature is a measure of how much a line is curve



$r$ : radius of curvature  
 $\kappa = \frac{1}{r}$ : curvature

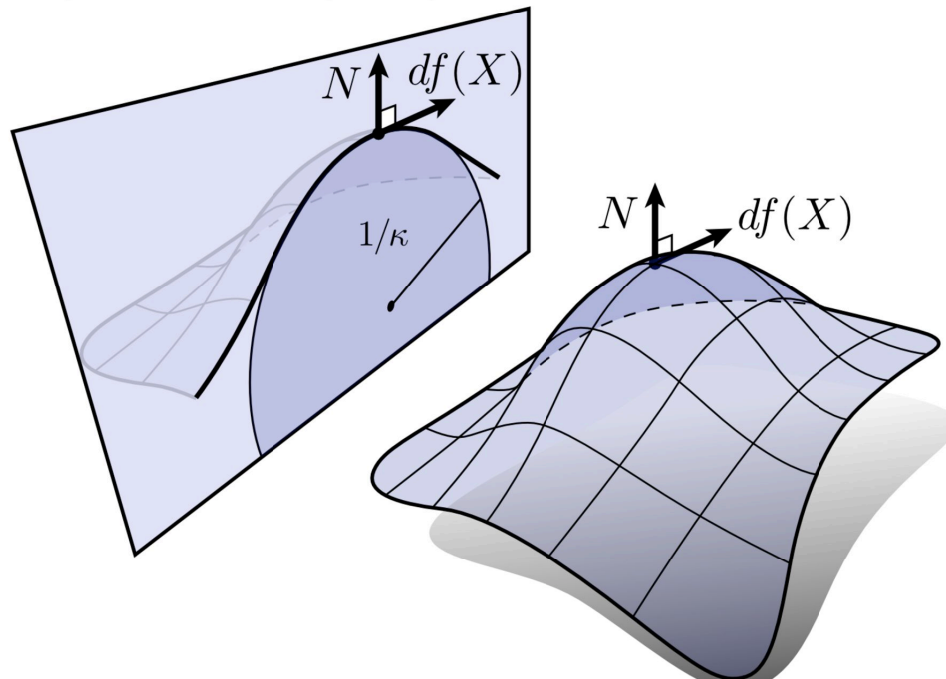
$x(t), y(t)$  a parametric curve  
 $\varphi$  tangential angle  
 $s$  arc length

$$\kappa \equiv \frac{d\varphi}{ds} = \frac{d\varphi/dt}{ds/dt} = \frac{d\varphi/dt}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}} = \frac{d\varphi/dt}{\sqrt{x'^2 + y'^2}}$$

$$\kappa = \frac{x' y'' - y' x''}{(x'^2 + y'^2)^{3/2}}$$

# Curvature on a surface

- ❖ Given the normal at point  $p$  and a tangent direction  $\theta$
- ❖ The curvature along  $\theta$  is the 2D curvature of the intersection between the plane and the surface



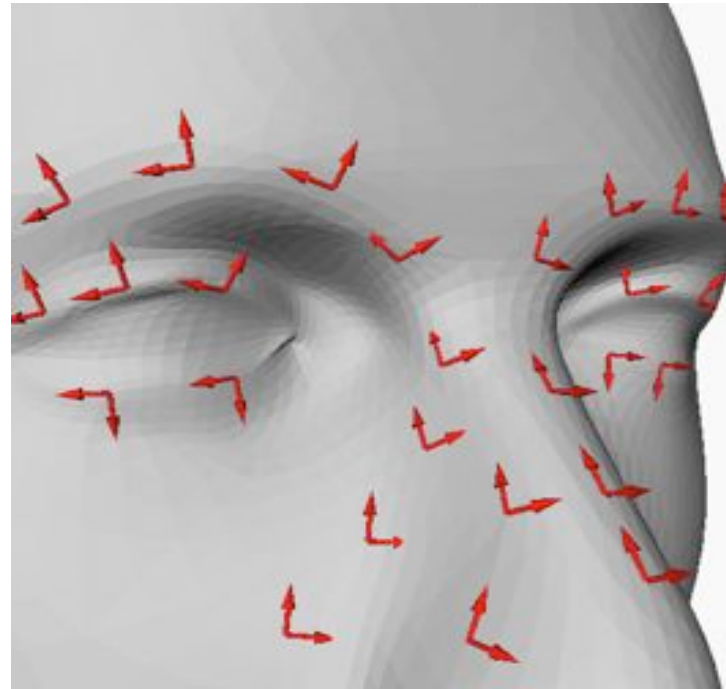
# Curvatures

- ❖ A curvature for each direction
- ❖ Take the two directions for which curvature is max and min

$\kappa_1, \kappa_2$  *principal curvatures*

$e_1, e_2$  *principal directions*

- ❖ the directions of max and min curvature are orthogonal



[Meyer02]

# Gaussian and Mean curvature

- ❖ Gaussian curvature: the product of principal curvatures

$$\kappa_G \equiv K \equiv \kappa_1 \cdot \kappa_2$$

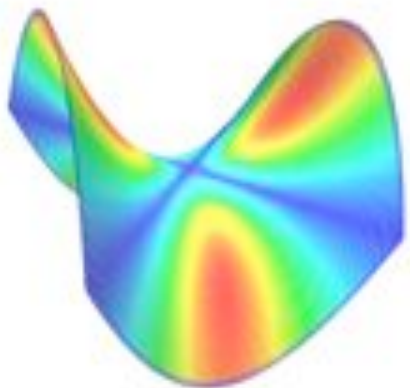
- ❖ Mean curvature: the average of principal curvatures

$$\bar{\kappa} \equiv H \equiv \frac{\kappa_1 + \kappa_2}{2}$$

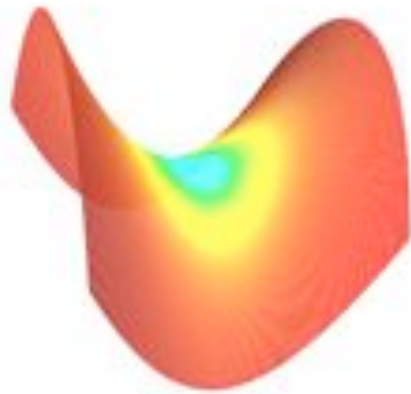
# Examples..

- ❖ Red:low → red:high (not in the same scale)

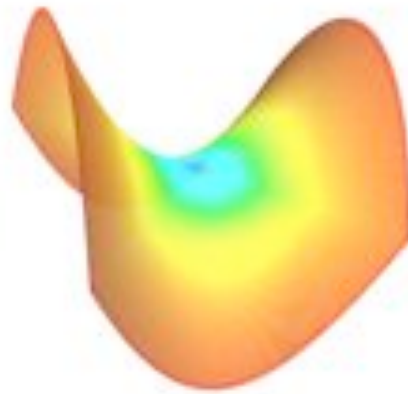
mean



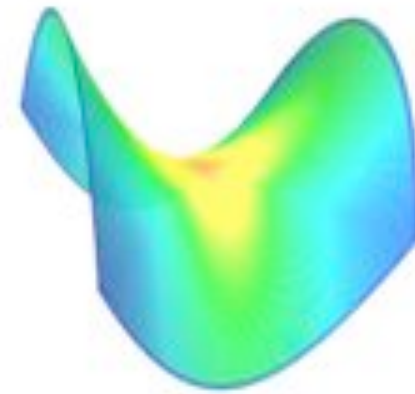
gaussian



min



max



[Meyer02]

# Gaussian Curvature

- ❖ Gaussian curvature is an intrinsic property
  - ❖ It can be computed by a bidimensional inhabitant of the surface by walking around a fixed point  $\mathbf{p}$  and keeping at a distance  $r$ .

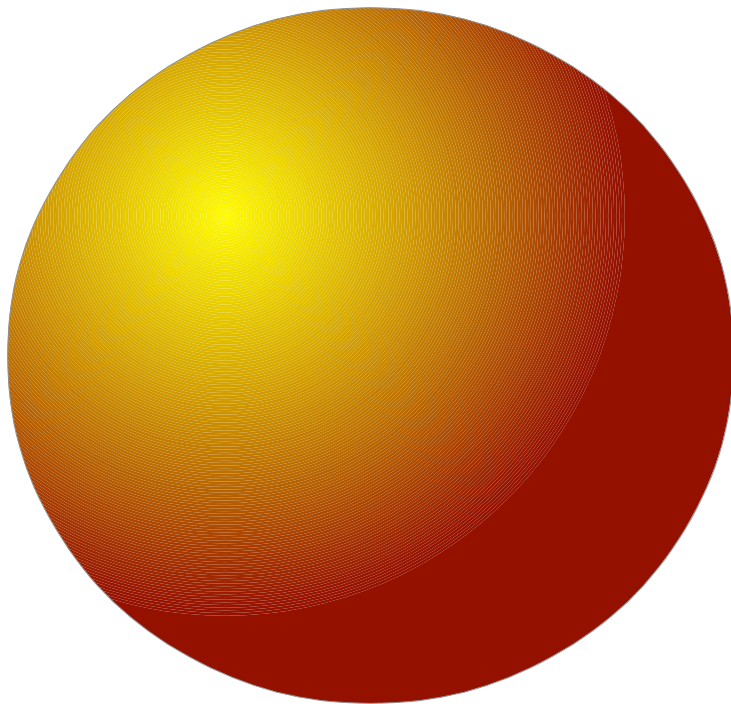
$$\kappa_G = \lim_{r \rightarrow 0} (2\pi r - C(r)) \cdot \frac{3}{\pi r^3}$$

with  $C(r)$  the distance walked

- ❖ Delevopable surfaces: surfaces whose Gaussian curvature is 0 everywhere

# Gaussian Curvature

$$\kappa_G > 0$$

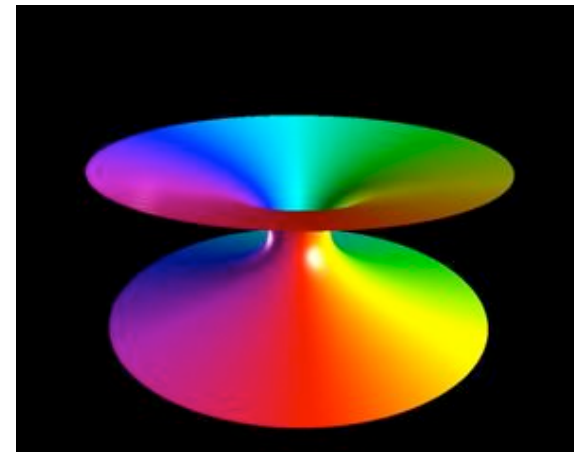
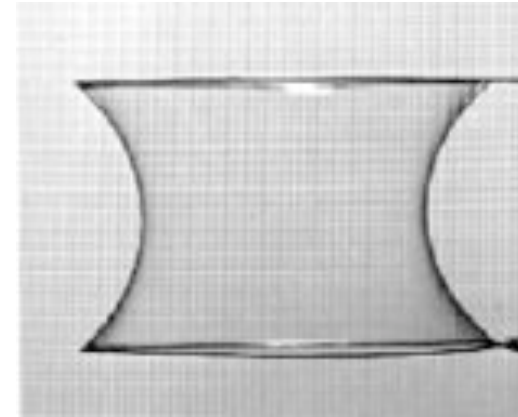


$$\kappa_G < 0$$



# Mean Curvature

- ❖ Divergence of the surface normal
  - ❖ divergence is an operator that measures a vector field's tendency to originate from or converge upon a given point
- ❖ Minimal surface and minimal area surfaces
  - ❖ A surface is *minimal* when its mean curvature is 0 everywhere
  - ❖ All minimal area surfaces have mean curvature 0
- ❖ The surface tension of an interface, like a soap bubble, is proportional to its mean curvature





# Mean Curvature

- ❖ Let  $A$  be the area of a disk around  $p$ . The mean curvature

$$2\bar{\kappa} n = \lim_{\text{diam}(A) \rightarrow 0} \frac{\nabla A}{A}$$

- ❖ the mean derivative is (twice) the divergence of the normal

$$2\bar{\kappa} = \nabla \cdot n = \frac{\partial}{\partial x} n + \frac{\partial}{\partial y} n + \frac{\partial}{\partial z} n$$

# Gaussian curvature on a triangle mesh

❖ It's the *angle defect* over the area

$$\kappa_G(v_i) = \frac{1}{3A} \left( 2\pi - \sum_{t_j \text{ adj } v_i} \theta_j \right)$$

❖ **Gauss-Bonnet Theorem:** The integral of the Gaussian Curvature on a closed surface depends on the Euler number

$$\int_S \kappa_G = 2\pi \chi$$

# Example data structure

- ❖ Simplest
- ❖ List of triangles:
  - ❖ For each triangle store its coords.
  - ❖
  - ❖ 1.  $(3,-2,5), (3,6,2), (-6,2,4)$
  - ❖ 2.  $(2,2,4), (0,-1,-2), (9,4,0)$
  - ❖ 3.  $(1,2,-2), (3,6,2), (-4,-5,1)$
  - ❖ 4.  $(-8,2,7), (-2,3,9), (1,2,-2)$
- ❖ How to find any adjacency?
- ❖ Does it store FV?

# Example data structure

- ❖ Slightly better
- ❖ List of unique vertices with indexed faces
  - ❖ Storing the FV relation

- ❖ Vertices:

- ❖ 1. (-1.0, -1.0, -1.0)
- ❖ 2. (-1.0, -1.0, 1.0)
- ❖ 3. (-1.0, 1.0, -1.0)
- ❖ 4. (-1, 1, 1.0)
- ❖ 5. (1.0, -1.0, -1.0)
- ❖ 6. (1.0, -1.0, 1.0)
- ❖ 7. (1.0, 1.0, -1.0)
- ❖ 8. (1.0, 1.0, 1.0)

- ❖ Faces:

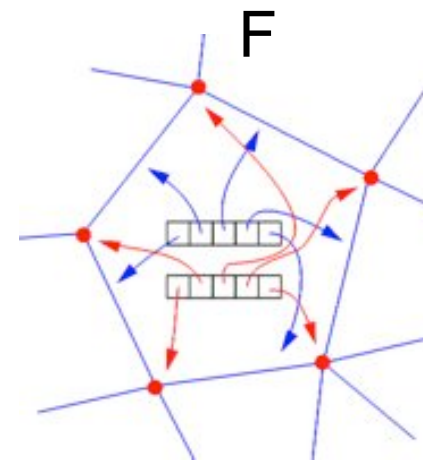
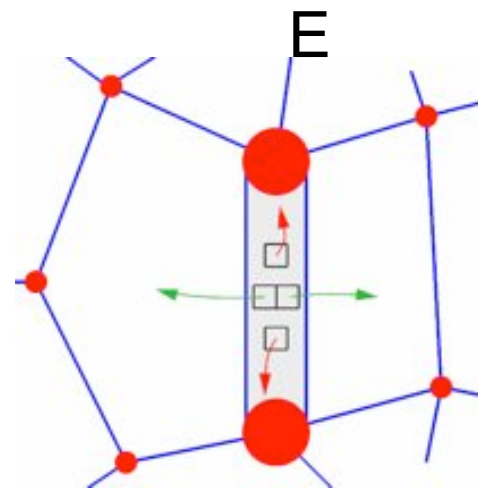
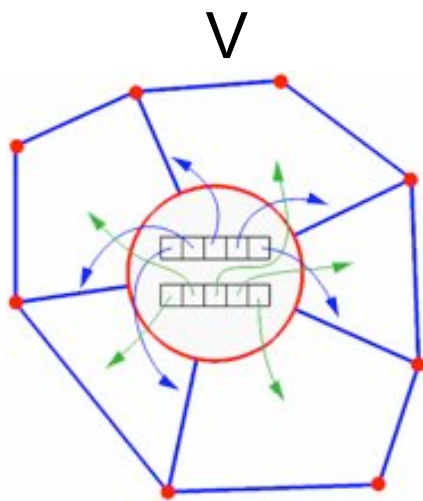
- ❖ 1. 1 2 4
- ❖ 2. 5 7 6
- ❖ 3. 1 5 2
- ❖ 4. 3 4 7
- ❖ 5. 1 7 5

# Example data structure

## ❖ Issue of Adjacency

### ❖ Vertex, Edge, and Face Structures

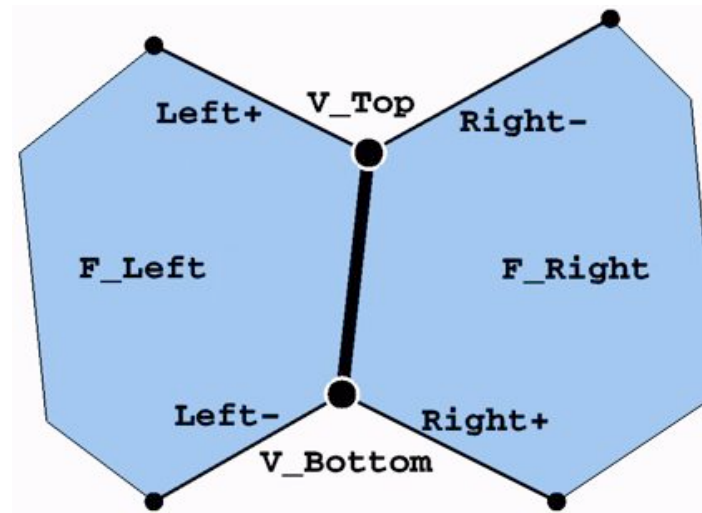
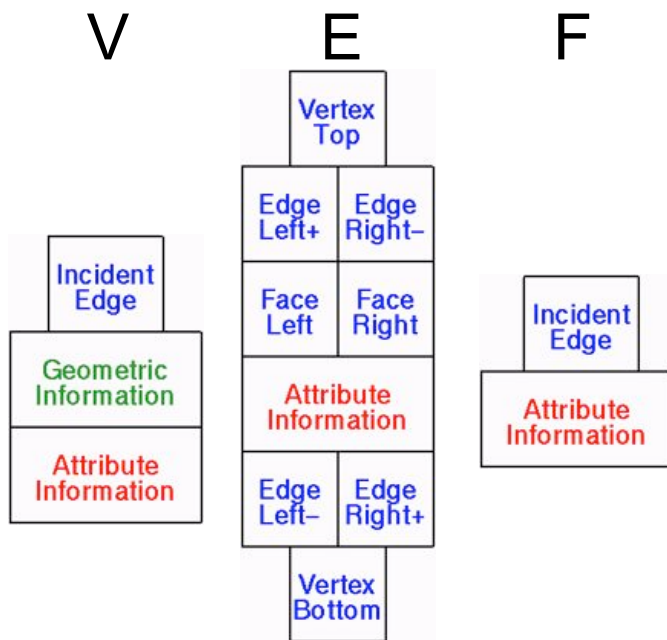
- ❖ Each element has list of pointers to all incident elements
- ❖ Queries depend only on local complexity of mesh!
- ❖ Slow! Big! Too much work to maintain!
- ❖ Data structures do not have fixed size



# Example data structure

## ❖ Winged edge

- ❖ Classical real smart structure
- ❖ Nice for generic polygonal meshes
- ❖ Used in many sw packages



# Winged Edge

- ❖ Winged edge
  - ❖ Compact
  - ❖ All the query requires some kind of "traversal"
  - ❖ Not fitted for rendering...

